

ANTI-INVARIANT RIEMANNIAN SUBMERSIONS

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ABSTRACT. We give a general Lie-theoretic construction for anti-invariant almost Hermitian Riemannian submersions, anti-invariant quaternion Riemannian submersions, anti-invariant para-Hermitian Riemannian submersions, anti-invariant para-quaternion Riemannian submersions, and anti-invariant octonion Riemannian submersions. This yields many compact Einstein examples.

1. INTRODUCTION

We begin by establishing some notational conventions.

1.1. Riemannian submersions. Let M and N be smooth manifolds of dimension m and n , respectively, and let $\pi : M \rightarrow N$ be a smooth map. We say that π is a *submersion* if π_* is a surjective map from the tangent space $T_P M$ to the tangent space $T_{\pi P} N$ for every point P of M . Let g_M and g_N be Riemannian metrics on M and N . If $\pi : M \rightarrow N$ is a submersion, then the *vertical distribution* is the kernel of π_* and the *horizontal distribution* \mathcal{H} is \mathcal{V}^\perp . We may then decompose $TM = \mathcal{V} \oplus \mathcal{H}$. We say that π is a *Riemannian submersion* if π_* is an isometry from \mathcal{H}_P to $T_{\pi P} N$ for every point P of M . We refer to O'Neill [14] for further details concerning the geometry of Riemannian submersions. If g_M and g_N are pseudo-Riemannian metrics, we impose in addition the condition that the restriction of g_M to \mathcal{V} is non-degenerate to ensure that $\mathcal{V} \cap \mathcal{H} = \{0\}$. This gives rise to the notation of a *pseudo-Riemannian submersion*.

1.2. Hermitian geometry. An endomorphism J of TM is said to define an *almost complex structure* on M if $J^2 = -\text{id}$, i.e. J gives a complex structure to $T_P M$ for every point P of M . We complexify the tangent bundle and let

$$T^{1,0} := \{X \in TM \otimes_{\mathbb{R}} \mathbb{C} : JX = \sqrt{-1}X\}.$$

One says J is *integrable* if $T^{1,0}$ is integrable, i.e. X, Y belong to $C^\infty(T^{1,0})$ implies the complex Lie bracket $[X, Y]$ also belongs to $C^\infty(T^{1,0})$. The Newlander-Nirenberg Theorem [13] is the analogue in the complex setting of the Frobenius theorem in the real setting; J is integrable if and only if it arises from an underlying holomorphic structure on N . The Riemannian metric g_M is said to be *almost Hermitian* if $J^*g_M = g_M$, i.e. if $g_M(JX, JY) = g_M(X, Y)$ for all tangent vectors $X, Y \in T_P M$ and all points P of M ; the triple (M, g_M, J) is then said to be an *almost Hermitian manifold*; in the pseudo-Riemannian setting one obtains the notion of *almost pseudo-Hermitian manifold* similarly. The notation *Hermitian* or *pseudo-Hermitian* is used if the structure J is integrable.

Let (M, g_M, J) be an almost Hermitian manifold and let π be a Riemannian submersion from (M, g_M) to (N, g_N) . Following the seminal work of Şahin [17, 18],

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one says that π is an *anti-invariant almost Hermitian Riemannian submersion* if

$$J\{\mathcal{V}\} \subset \mathcal{H}.$$

If $J\{\mathcal{V}\} = \mathcal{H}$, then π is said to be *Lagrangian*. There have been a number of subsequent papers in this subject extending the work of Şahin [17, 18]; we shall cite just a few representative examples. Lee et al. [11] examined the geometry of anti-invariant Hermitian submersions from a Kähler manifold onto a Riemannian manifold in relation to the Einstein condition and examined when the submersions were Clairant submersions. Ali and Fatima [1] examined the nearly Kähler setting. We also refer to related work of Ali and Fatima [2], of Beri et al. [5], and of Murthan and Küpeli-Erken [12].

1.3. Quaternion geometry. We shall restrict to flat quaternion structures as this is sufficient for our purposes. The *quaternion algebra* $\mathbb{Q} := \mathbb{R}^4 = \text{Span}\{e_0, e_1, e_2, e_3\}$ is defined by the relations:

$$\begin{array}{c|c|c|c|c} & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_1 & e_2 & e_3 \\ e_1 & e_1 & -e_0 & e_3 & -e_2 \\ e_2 & e_2 & -e_3 & -e_0 & e_1 \\ e_3 & e_3 & e_2 & -e_1 & -e_0 \end{array}. \quad (1.a)$$

One says that $x \in \mathbb{Q}$ is an *imaginary quaternion* if $x \in \text{Span}\{e_1, e_2, e_3\}$. A *flat quaternion structure* on a manifold M is a unital action of \mathbb{Q} on TM . If x is a unit length purely imaginary quaternion, then $\xi \rightarrow x \cdot \xi$ defines an almost complex structure on M . If g is a Riemannian metric on M , we shall assume in addition that $\|x \cdot \xi\| = \|x\| \cdot \|\xi\|$ for any quaternion x and any tangent vector ξ . Let $\pi : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian submersion. Then one says π is an *anti-invariant quaternion Riemannian submersion* if $x \cdot \mathcal{V} \subset \mathcal{H}$ for any purely imaginary quaternion x . We have assumed that the roles of $\{e_1, e_2, e_3\}$ are globally defined (i.e. the structure is flat); we refer to Alekseevsky and Marchiafava [3] for a discussion of the more general setting. Anti-invariant quaternion Riemannian submersions have been studied by K. Park [16].

1.4. Para-Hermitian geometry. Instead of considering almost complex structures, one can consider para-complex structures. Let $\mathbb{C} := \mathbb{R}^2$ with the para-complex structure $Je_1 = e_2$ and $Je_2 = e_1$. Let (M, g_M) be a pseudo-Riemannian manifold of neutral signature (ℓ, ℓ) . If J is an endomorphism of M with $J^2 = \text{Id}$ such that $g_M(JX, JY) = -g_M(X, Y)$ for all $X, Y \in T_P M$ and all points P of M , then the triple (M, g_M, J) is said to be a *para-Hermitian manifold*. Let π be a pseudo-Riemannian submersion from (M, g_M) to (N, g_N) with $J\{\mathcal{V}\} \subset \mathcal{H}$. One then says π is an *anti-invariant para-Hermitian Riemannian submersion*; π is *Lagrangian para-Hermitian* if $J\{\mathcal{V}\} = \mathcal{H}$. Atceken [4] and Gündüzalp [7] examined this setting. Gündüzalp [8] also examined the anti-invariant almost product setting; in the interests of brevity we shall not treat this setting in this paper although our methods are clearly applicable.

1.5. Para-quaternion geometry. In place of the quaternion commutation relations given in Equation (1.a), one imposes the para-quaternion relations to define the para-quaternions $\tilde{\mathbb{Q}}$ by setting:

$$\begin{array}{c|c|c|c|c} & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_1 & e_2 & e_3 \\ e_1 & e_1 & -e_0 & e_3 & -e_2 \\ e_2 & e_2 & -e_3 & +e_0 & -e_1 \\ e_3 & e_3 & e_2 & +e_1 & +e_0 \end{array}.$$

If J_1 is Hermitian and if J_2 and J_3 are para-Hermitian, then one obtains the notion of a *para-quaternion manifold*. We refer to Ivanov and Zamkovoy [10] for further details. If $\pi : (M, g_M) \rightarrow (N, g_N)$ is a pseudo-Riemannian submersion and if $J_i\{\mathcal{V}\} \subset \mathcal{H}$ for $1 \leq i \leq 3$, then π is said to be an *anti-invariant para-quaternion Riemannian submersion*. To the best of our knowledge, there are no papers on such geometries.

1.6. Octonian geometry. The octonians \mathbb{O} arise from a non-associative and non-commutative bilinear multiplication on \mathbb{R}^8 . If $\{e_0, \dots, e_7\}$ is the standard basis for \mathbb{R}^8 , the multiplication is given by the following table (see Wikipedia [19]):

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The octonians satisfy the identity

$$\|x \cdot y\| = \|x\| \cdot \|y\| \text{ for all } x, y \in \mathbb{R}^8.$$

If $x \in \text{Span}\{e_1, \dots, e_7\}$, then x is said to be a *purely imaginary octonian*. Such an octonian satisfies $x \cdot y \perp y$ for any $y \in \mathbb{R}^8$. Let (M, g) be a Riemannian manifold. A *flat octonian* structure on a Riemannian manifold (M, g) is a unital octonian action on TM such that $\|x \cdot \xi\| = \|x\| \cdot \|\xi\|$ for any octonian x and any tangent vector ξ . If π is a Riemannian submersion from (M, g) to (N, h) , then we say that π is *anti-invariant octonian* if $x \cdot \mathcal{V} \perp \mathcal{V}$ for any purely imaginary octonian $x \in \mathbb{R}^7$. To the best of our knowledge, there are no papers dealing with anti-invariant octonian Riemannian submersions.

1.7. Outline of the paper. In Section 2, we will use Lie theoretic methods to construct examples of anti-invariant almost Hermitian Riemannian submersions, of anti-invariant quaternion Riemannian submersions, of anti-invariant para-Hermitian Riemannian submersions, and of anti-invariant para-quaternion Riemannian submersions. In Section 3, we will discuss some examples which arise from this construction. We conclude in Section 4 by presenting a different family of examples (including an anti-invariant octonian Riemannian submersion) relating to the Hopf fibration where the total space is not a Lie group. It is our hope that having a rich family of examples will inform further investigations in this field.

2. A LIE-THEORETIC CONSTRUCTION

Let H be a closed and connected subgroup of an even dimensional Lie group G . Let \mathfrak{h} and \mathfrak{g} be the associated Lie algebras, respectively. Let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{g}}$ be a non-degenerate symmetric bilinear form on \mathfrak{g} which is invariant under the adjoint action of H and whose restriction to \mathfrak{h} is non-degenerate as well. We use $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ to decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ as an orthogonal direct sum. The inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ defines a left-invariant pseudo-Riemannian metric on G and, since the inner product is invariant under the adjoint action of H , the restriction of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ to \mathfrak{h}^{\perp} defines a G -invariant pseudo-Riemannian metric on the coset manifold G/H so that the natural projection $\pi : G \rightarrow G/H$ is a pseudo-Riemannian submersion.

2.1. Complex geometry. Assume $\langle \cdot, \cdot \rangle$ is positive definite. Let J be a Hermitian complex structure on \mathfrak{g} ; J induces a left-invariant Hermitian almost complex structure on G . Assume that $J\{\mathfrak{h}\} \subset \mathfrak{h}^\perp$. Then $\pi : G \rightarrow G/H$ is an anti-invariant almost Hermitian Riemannian submersion; π is Lagrangian if and only if $2 \dim\{\mathfrak{h}\} = \dim\{\mathfrak{g}\}$. More generally, if $\langle \cdot, \cdot \rangle$ is only assumed to be a non-degenerate inner product and if the restriction to \mathfrak{h} is assumed to be non-degenerate, then we obtain an anti-invariant almost pseudo-Hermitian Riemannian submersion.

2.2. Quaternion geometry. Assume $\langle \cdot, \cdot \rangle$ is positive definite. Assume given a Hermitian quaternion structure on \mathfrak{g} such that $x \cdot \mathfrak{h} \subset \mathfrak{h}^\perp$ for any purely imaginary quaternion x . Then $\pi : G \rightarrow G/H$ is an anti-invariant quaternion Riemannian submersion.

2.3. Para complex geometry. Assume $\langle \cdot, \cdot \rangle$ has neutral signature and that $\langle \cdot, \cdot \rangle$ is non-degenerate on \mathfrak{h} . Let J be a Hermitian para-complex structure on \mathfrak{g} with $J\{\mathfrak{h}\} \subset \mathfrak{h}^\perp$. Then $\pi : G \rightarrow G/H$ is an anti-invariant para-Hermitian Riemannian submersion; π is Lagrangian if and only if $2 \dim\{\mathfrak{h}\} = \dim\{\mathfrak{g}\}$.

2.4. Para-quaternion geometry. Assume $\langle \cdot, \cdot \rangle$ has neutral signature and that $\langle \cdot, \cdot \rangle$ is non-degenerate on \mathfrak{h} . Assume given a Hermitian para-quaternion structure on \mathfrak{g} such that $x \cdot \mathfrak{h} \subset \mathfrak{h}^\perp$ for any purely imaginary para-quaternion x . Then $\pi : G \rightarrow G/H$ is an anti-invariant para-quaternion Riemannian submersion.

3. EXAMPLES

In this section, we present examples of anti-invariant Riemannian submersions where the total space is a Lie group G and the base space is a homogeneous space upon which G acts transitively by isometries; $H \subset G$ is the isotropy subgroup of the action. Example 3.1 and Example 3.2 are flat geometries. Example 3.3 arises from the Hopf fibration $S^1 \rightarrow S^3 \rightarrow S^2$. In Example 3.4, the total space will be $(S^3)^\nu$. We will take product metrics and if the metric on S^3 is the usual round metric, these examples will be Einstein. In Example 3.5, we take $G = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R})$ to construct negative curvature examples.

3.1. Abelian examples.

Example 3.1. Let $G = \mathbb{R}^m$ and let $H = \mathbb{R}^n \oplus 0 \subset G$ for $n < m$. We identify G/H with $0 \oplus \mathbb{R}^{m-n}$ and π with projection on the last $m - n$ coordinates.

- (1) Take the standard Euclidean inner product on G to obtain a bi-invariant Riemannian metric so that π is a Riemannian submersion.
 - (a) Suppose $m = 2\ell$ and $n = \ell$. Identify $G = \mathbb{C}^\ell$ so that H corresponds to the purely real vectors in \mathbb{C}^ℓ . We identify \mathfrak{g} with G and \mathfrak{h} with H . Then $\sqrt{-1}\mathfrak{h} \perp \mathfrak{h}$ and we obtain a Lagrangian Hermitian Riemannian submersion; the almost complex structure corresponds to scalar multiplication by $\sqrt{-1}$ and is integrable.
 - (b) Assume $m = 4\ell$ and $n = \ell$. Identify $G = \mathbb{Q}^\ell$ so that H corresponds to the purely real vectors in \mathbb{Q}^ℓ . Then $x \cdot \mathfrak{h} \perp \mathfrak{h}$ if x is a purely imaginary quaternion and we obtain a Riemannian submersion which is anti-invariant quaternion.
 - (c) Assume $m = 8\ell$ and $n = \ell$. Identify $G = \mathbb{O}^\ell$ so that H corresponds to the purely real vectors in \mathbb{O}^ℓ . Then $x \cdot \mathfrak{h} \perp \mathfrak{h}$ if x is a purely imaginary octonian and we obtain a Riemannian submersion which is anti-invariant octonian.

- (2) Let $m = 2\ell$ and $n = \ell$. Identify G with $\tilde{\mathbb{C}}^\ell$ so that $H = \mathbb{R}^\ell$ corresponds to the purely real para-complex vectors. More specifically, we take a basis $\{e_i, f_i\}$ for $\mathbb{R}^{2\ell}$ where $H = \text{Span}\{e_i\}$. Set

$$\langle e_i, e_i \rangle = 1, \langle f_i, f_i \rangle = -1, \langle e_i, f_j \rangle = 0, \tilde{J}e_i = f_i, \tilde{J}f_i = e_i.$$

We obtain a Riemannian submersion which is Lagrangian para-Hermitian. By taking a different inner product $\langle e_i, e_i \rangle = -\langle f_i, f_i \rangle = \epsilon_i$ for $\epsilon_i = \pm 1$, we can ensure that the base has arbitrary signature.

- (3) Let $m = 4\ell$ and $n = \ell$. Identify $G = \tilde{\mathbb{Q}}^\ell$ so that $H = \mathbb{R}^\ell$ corresponds to the purely real vectors in G . We obtain a Riemannian submersion which is anti-invariant para-quaternion.

The total space $G = \mathbb{R}^m$ is non-compact in Example 3.1. We compactify by dividing by an integer lattice.

Example 3.2. Let \mathbb{Z}^k be the integer lattice in \mathbb{R}^k and let $\mathbb{T}^k := \mathbb{R}^k/\mathbb{Z}^k$ be the k -dimensional torus $S^1 \times \cdots \times S^1$ with the flat product metric. Let $G = \mathbb{T}^m$ and let $H = \mathbb{T}^n$. We can repeat the construction of Example 3.1 to obtain examples which are compact.

3.2. The Hopf fibration. Example 3.1 and Example 3.2 are flat. We can use the Hopf fibration to construct examples which are not flat. We identify \mathbb{R}^4 with the quaternions \mathbb{Q} ; this identifies S^3 with the unit quaternions and gives S^3 a Lie group structure. Let

$$e_1(x) = x \cdot i, \quad e_2(x) = x \cdot j, \quad e_3(x) = x \cdot k. \quad (3.a)$$

This is then a basis for the Lie algebra \mathfrak{g} of left-invariant vector fields on S^3 and

$$[e_1, e_2] = -2e_3, \quad [e_2, e_3] = -2e_1, \quad [e_3, e_1] = -2e_2.$$

Every 1-dimensional Lie subalgebra of S^3 corresponds to a 1-dimensional compact Abelian subgroup S^1 of S^3 . Let $G = S^1 \times S^3$ and let e_0 generate the Lie algebra of S^1 so that $\mathfrak{g} = \text{Span}\{e_0, e_1, e_2, e_3\}$. If $\epsilon \neq 0$, define

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } i = j = 2 \\ 1 & \text{if } i = j = 3 \\ \epsilon & \text{if } i = j = 0 \\ \epsilon & \text{if } i = j = 1 \\ 0 & \text{otherwise} \end{cases}. \quad (3.b)$$

These metrics are among the metrics first introduced by Hitchin [9] in his study of harmonic spinors and are Kaluza-Klein metrics. Let

$$\begin{aligned} Je_0 &= e_1, & Je_1 &= -e_0, & Je_2 &= e_3, & Je_3 &= -e_2, \\ \tilde{J}e_0 &= e_2, & \tilde{J}e_2 &= -e_0, & \tilde{J}e_3 &= e_1, & \tilde{J}e_1 &= -e_3, \\ \tilde{J}e_0 &= e_2, & \tilde{J}e_2 &= e_0, & \tilde{J}e_1 &= -e_3, & \tilde{J}e_3 &= -e_1. \end{aligned} \quad (3.c)$$

Then J is an integrable Hermitian complex structure on $S^1 \times S^3$ for any $\epsilon \neq 0$. If $\epsilon = 1$, then $\{1, J, \tilde{J}, J\tilde{J}\}$ is a Hermitian quaternion structure on $S^1 \times S^3$. If $\epsilon = -1$, then \tilde{J} is a para-Hermitian para-complex structure and $\{1, J, \tilde{J}, J\tilde{J}\}$ is a Hermitian para-quaternion structure on $S^1 \times S^3$. If $\epsilon = 1$, then $\langle \cdot, \cdot \rangle$ is bi-invariant. If $\epsilon \neq 1$, then $\langle \cdot, \cdot \rangle$ is right invariant under the 2-dimensional Lie subgroup H with $\mathfrak{h} = \text{Span}\{e_0, e_1\}$ but is not bi-invariant.

Example 3.3. Let $G = S^1 \times S^3$. Let \mathfrak{h} be the Lie sub-algebra of a closed subgroup H of G . Adopt the notation of Equation (3.b) and Equation (3.c).

- (1) Let $\mathfrak{h} = \text{Span}\{e_0, e_1\}$.

- (a) Let $\epsilon = 1$. Then $G/H = S^2$ is the sphere of radius 2 in \mathbb{R}^3 and has constant sectional curvature $\frac{1}{4}$. We use \tilde{J} to obtain a Lagrangian Hermitian Riemannian submersion. The fibers of the submersion are minimal, not totally geodesic, and the horizontal distribution is not integrable (see Park [15]).
- (b) Let $\epsilon = -1$. Then $G/H = S^2$. We use \tilde{J} to obtain a Lagrangian para-Hermitian Riemannian submersion.
- (2) If $\mathfrak{h} = \text{Span}\{e_0\}$, set $B = S^3$. If $\mathfrak{h} = \text{Span}(e_1)$, set $B = S^1 \times S^3$.
 - (a) Let $\epsilon \neq 0$ be arbitrary. We use J to obtain an anti-invariant Hermitian Riemannian submersion from G to B .
 - (b) Let $\epsilon = +1$. We use J and \tilde{J} to identify $\mathfrak{g} = \mathbb{Q}$ with the quaternions to obtain an anti-invariant quaternion Riemannian submersion from G to B .
 - (c) Let $\epsilon = -1$. We use \tilde{J} to obtain an anti-invariant Hermitian Riemannian submersion from G to B .
 - (d) Let $\epsilon = -1$. We use J and \tilde{J} to identify $\mathfrak{g} = \mathbb{Q}$ with the para-quaternions to obtain an anti-invariant para-quaternion Riemannian submersion from G to B .

3.3. Einstein geometry.

Example 3.4. Let g_{S^3} be the standard round metric on S^3 defined by $\langle e_i, e_j \rangle = \delta_{ij}$. Let $G = (S^3)^\nu = S^3 \times \cdots \times S^3$. We take a product metric on G where the metric on each factor is $\pm g_{S^3}$; thus this metric is bi-invariant. Let H be a closed subgroup of G and let $\pi : G \rightarrow G/H$ be the associated Riemannian submersion.

- (1) Let $G = S^3 \times S^3$ and $\mathfrak{g} = \text{Span}\{e_1, e_2, e_3, f_1, f_2, f_3\}$.
 - (a) Let $g_G = g_{S^3} \oplus g_{S^3}$ be the standard bi-invariant Einstein metric on $S^3 \times S^3$.
 - (i) Let $\mathfrak{h} = \text{Span}\{e_2, f_2\}$. Let $Je_1 = f_1$, $Jf_1 = -e_1$, $Je_2 = e_3$, $Je_3 = -e_2$, $Jf_2 = f_3$, $Jf_3 = -f_2$. This almost complex structure is integrable and using J yields an anti-invariant Hermitian Riemannian submersion from $S^3 \times S^3$ to $S^2 \times S^2$.
 - (ii) Let $\mathfrak{h} = \text{Span}\{e_1, e_2, e_3\}$. Let $Je_i = f_i$ and $Jf_i = -e_i$ for $1 \leq i \leq 3$. This almost complex structure is not integrable. Using J yields an anti-invariant almost Hermitian Riemannian submersion from $S^3 \times S^3$ to S^3 .
 - (b) Let $g_G = g_{S^3} \oplus -g_{S^3}$ be the standard bi-invariant neutral signature metric on $S^3 \times S^3$.
 - (i) Let $\mathfrak{h} = \text{Span}\{e_1, f_1\}$. Let $\tilde{J}e_1 = f_2$, $\tilde{J}f_2 = e_1$, $\tilde{J}f_1 = e_2$, $\tilde{J}e_2 = f_1$, $\tilde{J}e_3 = f_3$, and $\tilde{J}f_3 = e_3$. Using \tilde{J} yields an anti-invariant para-Hermitian Riemannian submersion from $S^3 \times S^3$ to $S^2 \times S^2$.
 - (ii) Let $\mathfrak{h} = \text{Span}\{e_1, e_2, e_3\}$. Let $\tilde{J}e_i = f_i$ and $\tilde{J}f_i = e_i$. Using \tilde{J} yields an anti-invariant para-Hermitian Riemannian submersion from $S^3 \times S^3$ to S^3 .
- (2) Let $G = (S^3)^4$, let $g_G = g_{S^3} \oplus g_{S^3} \oplus g_{S^3} \oplus g_{S^3}$, and let $\dim\{H\} \leq 3$.
 - (a) Identify \mathfrak{g} with \mathbb{Q}^3 in such a way that \mathfrak{h} is real and the action of \mathbb{Q} is Hermitian. Then π is an anti-invariant quaternion.
 - (b) Identify \mathfrak{g} with $\tilde{\mathbb{Q}}^3$ in such a way that \mathfrak{h} is real and the action of $\tilde{\mathbb{Q}}$ is para-Hermitian. Then π is an anti-invariant para-quaternion Riemannian submersion.

- (3) Let $G = (S^3)^8$, let $g_G = g_{S^3} \oplus \dots \oplus g_{S^3}$, and let $\dim\{H\} \leq 7$. Identify \mathfrak{g} with \mathbb{O}^3 in such a way that \mathfrak{h} is real and the action of \mathbb{O} is Hermitian. Then π is an anti-invariant octonian Riemannian submersion.

3.4. Negative curvature. Our previous examples have, for the most part, involved the Lie group S^3 and the Hopf fibration $S^3 \rightarrow S^2$. We now turn to the negative curvature dual. Let $\mathbb{H}^2(0, 2)$ be the hyperbolic plane with a Riemannian metric of constant sectional curvature $-\frac{1}{4}$ and let $\mathbb{H}^2(1, 1)$ be the Lorentzian analogue. We recall some facts about the Lie group $\mathrm{SL}(2, \mathbb{R})$ and refer to Section 6.8 of Gilkey, Park, and Vázquez-Lorenzo [6] – there are, of course, many excellent references. $\mathrm{SL}(2, \mathbb{R})$ is a 3-dimensional Lie group and the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the vector space of trace free 2×2 real matrices. The canonical basis for $\mathfrak{sl}(2, \mathbb{R})$ is

$$f_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The bracket relations then take the form

$$[f_1, f_2] = 2f_3, \quad [f_2, f_3] = -2f_1, \quad [f_3, f_1] = 2f_2.$$

The Lie algebra \mathfrak{s}^3 of S^3 is the Lie algebra of the special unitary group $SU(2)$ in positive definite signature and the Lie algebra of $\mathfrak{sl}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$ is the Lie algebra of the special unitary group $SU(1, 1)$ in indefinite signature; the two are related by complexification. Let $\mathrm{ad}(\xi) : \eta \rightarrow [\xi, \eta]$ be the adjoint action and let $K(\xi, \eta) := \mathrm{Tr}\{\mathrm{ad}(\xi) \mathrm{ad}(\eta)\}$ be the Killing form. One then has

$$K(f_i, f_j) = \begin{cases} -8 & \text{if } i = j = 1 \\ +8 & \text{if } i = j = 2 \\ +8 & \text{if } i = j = 3 \\ 0 & \text{otherwise} \end{cases}.$$

There is no bi-invariant Riemannian metric on $\mathrm{SL}(2, \mathbb{R})$. However, $\frac{1}{8}K$ is a bi-invariant Lorentzian metric on $\mathrm{SL}(2, \mathbb{R})$. Let

$$\begin{aligned} \sigma_1(x) &:= \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}, & \sigma_2(x) &:= \begin{pmatrix} \cosh(x) & \sinh(x) \\ \sinh(x) & \cosh(x) \end{pmatrix}, \\ \sigma_3(x) &:= \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}. \end{aligned}$$

These define closed Abelian Lie sub-groups H_i of $\mathrm{SL}(2, \mathbb{R})$ whose associated Lie-algebras are spanned by f_i . The natural coset spaces $\mathrm{SL}(2, \mathbb{R})/H_i$ have constant negative sectional curvature $-\frac{1}{4}$ and may be identified with $\mathbb{H}^2(0, 2)$ if $i = 1$ and $\mathbb{H}^2(1, 1)$ if $i = 2, 3$.

Let $G = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R})$. Let f_0 correspond to the Abelian factor. Define a bi-invariant neutral signature metric on G by setting:

$$\langle f_i, f_j \rangle = \begin{cases} -1 & \text{if } i = j = 0 \\ -1 & \text{if } i = j = 1 \\ +1 & \text{if } i = j = 2 \\ +1 & \text{if } i = j = 3 \\ 0 & \text{otherwise} \end{cases},$$

In analogy with Equation (3.b), we set:

$$\begin{aligned} Jf_0 &= f_1, & Jf_1 &= -f_0, & Jf_2 &= f_3, & Jf_3 &= -f_2, \\ \tilde{J}f_0 &= f_2, & \tilde{J}f_2 &= f_0, & \tilde{J}f_1 &= -f_3, & \tilde{J}f_3 &= -f_1. \end{aligned}$$

Then J is a Hermitian complex structure on G and \tilde{J} is a Hermitian para-complex structure on G ; J and \tilde{J} generate a Hermitian para-complex structure on G .

Example 3.5. Let $G = \mathbb{R} \times \mathrm{SL}(2, \mathbb{R})$, let H be a closed subgroup of G , and let π be the natural projection from G to G/H .

- (1) If $\mathfrak{h} = \mathrm{Span}\{f_0\}$, then π is an anti-invariant Hermitian, an anti-invariant para-Hermitian, and an anti-invariant para-quaternion Riemannian submersion from G to $\mathrm{SL}(2, \mathbb{R})$.
- (2) If $\mathfrak{h} = \mathrm{Span}\{f_1\}$, then π is an anti-invariant Hermitian, an anti-invariant para-Hermitian, and an anti-invariant para-quaternion Riemannian submersion from G to $\mathbb{R} \times \mathbb{H}^2(0, 2)$.
- (3) If $\mathfrak{h} = \mathrm{Span}\{f_2\}$, then π is an anti-invariant Hermitian, an anti-invariant para-Hermitian, and an anti-invariant para-quaternion Riemannian submersion from G to $\mathbb{R} \times \mathbb{RH}^2(1, 1)$.
- (4) If $\mathfrak{h} = \mathrm{Span}\{f_0, f_2\}$, then $\pi : G \rightarrow G/H$ is an anti-invariant Hermitian Riemannian submersion from G to $\mathbb{H}^2(1, 1)$.

4. EXAMPLES WHERE THE TOTAL SPACE IS NOT A LIE GROUP

In this section, we present examples where the total space is not a Lie group. We identify \mathbb{R}^{4k} with \mathbb{C}^{2k} to define an action of S^1 on S^{4k-1} ; the quotient S^{4k-1}/S^1 is complex projective space \mathbb{CP}^{2k-1} with a Fubini-Study metric of constant positive holomorphic sectional curvature. We identify $\mathbb{R}^{4\ell}$ with \mathbb{Q}^ℓ to define an action of S^3 on $S^{4\ell-1}$; the quotient $S^{4\ell-1}/S^3$ is quaternionic projective space $\mathbb{QP}^{\ell-1}$. Instead of taking the Euclidean inner product on \mathbb{R}^{2k} , we could take an indefinite signature metric. Let

$$\begin{aligned} \langle x, y \rangle &:= -x^1 y^1 - x^2 y^2 + x^3 y^3 + \cdots + x^{2\ell} y^{2\ell}, \\ \ll x, y \gg &:= -x^1 y^1 - \cdots - x^4 y^4 + x^5 y^5 + \cdots + x^{4\ell} y^{4\ell} \\ \tilde{S}^{2k-1} &:= \langle x, y \rangle = -1, \quad \tilde{S}^{4k-1} := \{x \in \mathbb{R}^{4\ell} : \ll x, x \gg = -1\}. \end{aligned} \tag{4.a}$$

The pseudo-spheres \tilde{S}^{2k-1} and \tilde{S}^{4k-1} inherit indefinite signature metrics of constant sectional curvature. The quotient \tilde{S}^{2k-1}/S^1 is the negative curvature dual of \mathbb{CP}^{k-1} and the quotient \tilde{S}^{4k-1}/S^3 is the negative curvature of \mathbb{QP}^{k-1} .

Let $m = 2k$ and $N = S^{2k-1}$ or $N = \tilde{S}^{2k-1}$, or let $m = 4k$ and $N = \tilde{S}^{4k-1}$. There is an orthogonal direct sum decomposition $T(\mathbb{R}^m)|_N = \nu \oplus T(N)$ where ν is the normal bundle. Let $M = S^1 \times N$ with the product metric. Since ν is a trivial line bundle, we have a natural isometry $\Xi : TM \approx M \times \mathbb{R}^n$. Let ∂_θ be the natural unit tangent vector field on S^1 . Let $x = (\theta, \Theta) \in M$. Then $\Xi(\theta, \Theta)\nu = \Theta$ and $\Xi(TN) = \Theta^\perp$.

Example 4.1. Let $\ell \geq 3$. Adopt the notation of Equation (4.a). Let J be complex multiplication by i on $TM = M \times \mathbb{C}^\ell$. Let $H = S^1$. Since $\dim\{\mathcal{V}\} = 1$, $J\mathcal{V} \perp \mathcal{V}$.

- (1) Let $M = S^1 \times S^{2\ell-1}$. Let H act on S^1 by complex multiplication and trivially on $S^{2\ell-1}$. Then $\pi : M \rightarrow S^{2\ell-1}$ is an anti-invariant Hermitian Riemannian submersion.
- (2) Let $M = S^1 \times S^{2\ell-1}$. Let H act on trivially on S^1 and by complex multiplication on $S^{2\ell-1}$. Then $\pi : M \rightarrow S^1 \times \mathbb{CP}^{\ell-1}$ is an anti-invariant Hermitian Riemannian submersion.
- (3) Let $M = S^1 \times \tilde{S}^{2\ell-1}$. Let H act on S^1 by complex multiplication and trivially on $\tilde{S}^{2\ell-1}$. Then $\pi : M \rightarrow \tilde{S}^{2\ell-1}$ is an anti-invariant Hermitian Riemannian submersion.
- (4) Let $M = S^1 \times \tilde{S}^{2\ell-1}$. Let H act on trivially on S^1 and by complex multiplication on $\tilde{S}^{2\ell-1}$. Then $\pi : M \rightarrow S^1 \times \widetilde{\mathbb{CP}}^{\ell-1}$ is an anti-invariant Hermitian Riemannian submersion.

We have taken $\ell \geq 3$ since the case $\ell = 2$ recovers the Hopf fibration $S^3 \rightarrow S^2$ or $S^3 \rightarrow \mathbb{H}^2$.

Example 4.2. Adopt the notation of Equation (4.a). Let $\ell \geq 2$. Let $H = S^1 \times S^1$. Let J be quaternion multiplication on $TM = M \times \mathbb{Q}^\ell$.

- (1) Use the product action to let H act on the first and on the second factor of $M = S^1 \times S^{4\ell-1}$. Let π be the associated Riemannian submersion from M to $\mathbb{CP}^{2\ell-1}$. Then $\mathcal{V}(\theta, \Theta) = \text{Span}\{\Theta, i \cdot \Theta\}$. Since $j \cdot \mathcal{V} \perp \mathcal{V}$, π is an anti-invariant Hermitian Riemannian submersion.
- (2) Use the product action to let H act on the first and on the second factor of $M = S^1 \times S^{4\ell-1}$. Let π be the associated Riemannian submersion from M to $\mathbb{CP}^{2\ell-1}$. Then $\mathcal{V}(\theta, \Theta) = \text{Span}\{\Theta, i \cdot \Theta\}$. Since $j \cdot \mathcal{V} \perp \mathcal{V}$, π is an anti-invariant Hermitian Riemannian submersion.

We have taken $\ell \geq 2$ since the case $\ell = 1$ recovers the Hopf fibration $S^3 \rightarrow S^2$ or $S^3 \rightarrow \mathbb{H}^2$.

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